Wigner time-delay distribution in chaotic cavities and freezing transition

Christophe Texier[1, 2] and Satya N. Majumdar[1]

[1] *Univ. Paris Sud ; CNRS ; LPTMS, UMR 8626 ; Bât. 100, Orsay F-91405, France*

[2] *Univ. Paris Sud ; CNRS ; LPS, UMR 8502 ; Bât. 510, Orsay F-91405, France*

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Using the joint distribution for proper time-delays of a chaotic cavity derived by Brouwer, Frahm & Beenakker [Phys. Rev. Lett. **78**, 4737 (1997)], we obtain, in the limit of large number of channels N, the large deviation function for the distribution of the Wigner time-delay (the sum of proper times) by a Coulomb gas method. We show that the existence of a power law tail originates from narrow resonance contributions, related to a (second order) freezing transition in the Coulomb gas.

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The study of scattering theory in chaotic or disordered quantum systems within the random matrix theory (RMT) has been a subject of intense research for many years. Though originated in nuclear physics (see the review [1]), it has major implications in condensed matter theory where it can be used to model electronic transport in mesoscopic (coherent) conductors [2, 3]. The dynamics of an electron of energy E is described through the $N \times N$ on-shell scattering matrix $\mathcal{S}(E)$, where N is the number of scattering channels. A useful concept that characterises the temporal aspects of the scattering process is time-delay [4, 5] undergone by an incident wave packet. This is captured by the Wigner-Smith time-delay matrix [6], $Q(E) \stackrel{\text{def}}{=} -\mathrm{i}\,\mathcal{S}(E)^\dagger \frac{\partial \mathcal{S}(E)}{\partial E}$ (with $\hbar = 1$), whose eigenvalues are the *proper* time-delays $\tau_1, \cdots, \tau_N$.

If the system is characterised by some complex dynamics, due to the presence of disorder or chaos, the statistical properties of time-delays exhibit interesting universal characteristics : the universality of the time-delay distribution for 1D-disordered quantum mechanics was demonstrated in [7] (see also [8–12], and [13] for 2D & 3D cases). The situation where the dynamics is chaotic has been extensively studied within RMT: the marginal law of *partial* time-delays [14], $\tilde{p}_N(\tau) = \frac{1}{N}\sum_a \langle\delta(\tau - \tilde{\tau}_a)\rangle$, was obtained for GUE symmetry (Dyson index $\beta = 2$) in [15, 16]. The time-delay distribution was obtained in the $N = 1$ case with $\beta \in \{1, 2, 4\}$ in [20]. Using the "alternative RMT" introduced in [21], Brouwer and coworkers succeeded in finding the joint distribution of the inverse proper time-delays $\gamma_k \equiv 1/\tau_k$ [22, 23] :

$$P(\gamma_1, \cdots, \gamma_N) \propto \prod_{i<j} |\gamma_i - \gamma_j|^\beta \prod_k \gamma_k^{\beta N/2} e^{-\frac{\beta}{2}\gamma_k} \qquad (1)$$

(the times are measured in units of the Heisenberg time $\tau_{\mathrm{H}} = 2\pi\hbar/\Delta$, where Δ is the mean level spacing). This measure, known as the Laguerre ensemble of random matrices, also corresponds to the distribution of the (positive) eigenvalues of Wishart matrices $X^\dagger X$, where the matrix X has size $N \times (2N - 1 + 2/\beta)$ with i.i.d. Gaussian matrix elements.

In this article we are interested in the Wigner time delay, defined as the sum of proper (or partial) [14] time delays $\tau_{\mathrm{W}} \stackrel{\text{def}}{=} \frac{1}{N}\operatorname{Tr}\{Q\} = \frac{1}{N}\sum_{a=1}^N \tau_a$. This quantity is of great interest due to its close relation to the density of states (DoS) of the *open system*, through the Krein-Friedel relation [17] : $\nu(E) = \frac{1}{2\pi}\operatorname{Tr}\{Q(E)\} = \frac{1}{2\pi} N\tau_{\mathrm{W}}$. The Wigner time delay (or related quantities such as injectance or emittance) is a central concept for studying charging effects, e.g. for mesoscopic capacitances [18, 20].

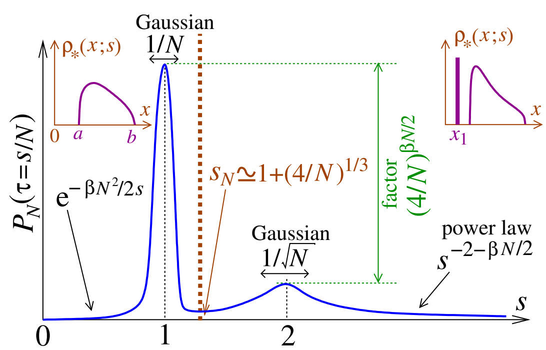

FIG. 1: (color online). *Sketch of the distribution of $s = N\tau_{\mathrm{W}}$. The dashed line at $s = s_N \simeq 1 + (4/N)^{1/3}$ separates the two phases of the Coulomb gas with densities represented in the small figures on the left and the right respectively.*

We denote by $P_N(\tau) \stackrel{\text{def}}{=} \langle\delta(\tau - \frac{1}{N}\sum_a \tau_a)\rangle$ the Wigner time-delay distribution. Despite the fact that the joint distribution of proper times is known already for 15 years, little is known about the distribution of τ_{W} for general N: it has been computed explicitly only for $N = 1$, $P_1(\tau) = \frac{(\beta/2)^{\beta/2}}{\Gamma(\beta/2)}\tau^{-2-\frac{\beta}{2}}\,e^{-\frac{\beta}{2\tau}}$ [20] and $N = 2$, $P_2(\tau) = \frac{\beta^{3\beta+2}\Gamma(3(\beta+1)/2)}{\Gamma(\beta+1)\Gamma(3\beta+2)}\tau^{-3(\beta+1)}\,U\left(\frac{\beta+1}{2}, 2(\beta+1); \beta/\tau\right)\,e^{-\beta/\tau}$ [24], where $U(a, b; z)$ is the confluent hypergeometric function. The distribution was conjectured to have a power law tail for large τ, $P_N(\tau) \sim \tau^{-2-\frac{\beta}{2}N}$ in [16] (for $\beta = 2$) by using the resonance picture allowing to identify the tails of $P_N(\tau)$ and $\tilde{p}_N(\tau)$ (for a heuristic argument using the relation to resonance width, cf. the review [25]). More recently, the first three cumulants of τ_{W} were derived by a generating function method [26]. However a full understanding of its distribution for

general N is still missing so far.

In this Letter, by analysing an underlying Coulomb gas we provide a complete description of $P_N(\tau)$ for large N and show that it has a rather rich behaviour including an interesting nonanalytic point which is a consequence of a freezing transition in the Coulomb gas. Limiting behaviours of $P_N(\tau)$ may be summarised as follows ($\tau_{\rm W}$ is measured in unit of $\tau_{\rm H}$) :

$$P_N(\tau) \sim \tau^{-\frac{3}{4}N^2\beta}\, e^{-\frac{N\beta}{2\tau}} \qquad \text{for } \tau \ll \frac{1}{N} \quad (2)$$

$$\sim \exp -\frac{N^4\beta}{8}\left(\tau-\frac{1}{N}\right)^2 \qquad \text{for } \tau \sim \frac{1}{N} \quad (3)$$

$$\sim N^{-\frac{\beta N}{2}} \exp -\frac{N^3\beta}{4}\left(\tau-\frac{2}{N}\right)^2 \quad \text{for } \tau \sim \frac{2}{N} \quad (4)$$

$$\sim \tau^{-2-\frac{N\beta}{2}} \qquad \text{for } \tau \gg \frac{1}{N}\,, \quad (5)$$

A sketch of the distribution is given in Fig. 1. The Gaussian form around $\tau \sim 1/N$ in (3) allows one to extract the mean time-delay and its variance. Reinstating $\tau_{\rm H}$, we obtain $\langle \tau_{\rm W}\rangle = \frac{\tau_{\rm H}}{N}$. Consequently, the mean DoS reads $\langle \nu(E)\rangle = N\langle\tau_{\rm W}\rangle/2\pi = 1/\Delta$ as expected. Similarly, the variance can be read off (3)

$$\mathrm{Var}(\tau_{\rm W}) \simeq \frac{4\tau_{\rm H}^2}{\beta N^4} \quad \text{i.e.} \quad \mathrm{Var}(\nu(E)) \simeq \frac{4}{\beta N^2\Delta^2}\,. \quad (6)$$

Eq. (6) agrees with the leading order of the result obtained in Ref. [26] $\mathrm{Var}(\tau_{\rm W}) = \frac{4\tau_{\rm H}^2}{(N+1)(N\beta-2)N^2}$. Note also that (5) coincides with the power law tail conjectured by Fyodorov and Sommers [16], $P_N(\tau) \sim \tau^{-2-\frac{\beta}{2}N}$.

Coulomb gas.– To derive our main results (2,3,4,5), we use the Coulomb gas method, originally introduced by Dyson [27]. Recently, this method has been suitably adopted and successfully used in a number of different contexts : e.g. the distribution of the conductance of chaotic cavities [28–30], or the quantum entanglement in a random bipartite state [31–33]. Our starting point is to rewrite the joint distribution (1) of the rescaled rates $x_i = \gamma_i/N$ as a Gibbs measure, $P(\gamma_1,\cdots,\gamma_N) \propto \exp -\frac{1}{2}\beta N^2\mathcal{E}[\rho]$, with the "energy" $\mathcal{E}[\rho]$ expressed as a functional of the density of the rescaled rates $\rho(x) = \frac{1}{N}\sum_{i=1}^N \delta(x-x_i)$. The energy reads

$$\mathcal{E}[\rho] = \int_0^\infty \mathrm{d}x\,(x-\ln x)\,\rho(x) - \int_0^\infty \mathrm{d}x\mathrm{d}x'\,\rho(x)\,\rho(x')\,\ln|x-x'| \quad (7)$$

The rescaled time-delay is $s = N\tau_{\rm W} = \sum_i \gamma_i^{-1}$ (i.e. the DoS of the cavity in appropriate units $s = \nu(E)\Delta$). In the limit $N\to\infty$, the density $\rho(x)$ may be treated as continuous and the distribution $P_N(\tau = s/N)$ can be derived via a saddle point method. The optimal (saddle point) distribution minimizes (7) with two constraints: normalisation $\int \mathrm{d}x\,\rho(x) = 1$ and $\int \frac{\mathrm{d}x}{x}\,\rho(x) = s$. This requires to minimize the "free energy" $\mathcal{F}[\rho] = \mathcal{E}[\rho] + \mu_0\big(\int \mathrm{d}x\,\rho(x) - 1\big) + \mu_1\big(\int \frac{\mathrm{d}x}{x}\,\rho(x) - s\big)$, where μ_0 and μ_1 are two Lagrange multipliers that enforce the two constraints (we neglect the subdominant contribution of entropy [34]). Setting the functional derivative $\frac{\delta\mathcal{F}}{\delta\rho(x_0)} = 0$ gives

$$\mu_0 + x_0 - \ln x_0 + \frac{\mu_1}{x_0} - 2\int_a^b \mathrm{d}x\,\rho(x)\,\ln|x-x_0| = 0\,. \quad (8)$$

where we assume that the optimal density has support over the interval $x_0 \in [a,b]$. Deriving once more with respect to x_0 gives

$$\frac{1}{2}\left(1 - \frac{1}{x_0} - \frac{\mu_1}{x_0^2}\right) = \fint_a^b \mathrm{d}x\,\frac{\rho(x)}{x_0-x}\,, \quad (9)$$

where $\fint$ represents the principal part. This equation expresses the force balance at equilibrium, for any charge at $x_0 \in [a,b]$, between the confining force $-V'_{\rm eff}(x)$ coming from the effective potential $V_{\rm eff}(x) = x - \ln x + \frac{\mu_1}{x}$ and the Coulomb repulsion force from other charges. We denote by $\rho_*(x;s)$ the solution of (9). The time-delay distribution then takes the scaling form

$$P_N(\tau) \underset{N\to\infty}{\sim} \exp -\frac{1}{2}\beta N^2 \Phi_-(N\tau)\,, \quad (10)$$

where the large deviation function is $\Phi_-(s) = \mathcal{E}[\rho_*(x;s)] - \mathcal{E}[\rho_*(x;1)]$ (note that when the two constraints are fulfilled, $\mathcal{F}[\rho_*] = \mathcal{E}[\rho_*]$). The term $\mathcal{E}[\rho_*(x;1)]$ emerges from the normalisation of (1), obtained by solving the same equation in the absence of the second constraint, i.e. for $\mu_1 = 0$, which we will show to coincide with $s = 1$. Using (8), we may rewrite the energy of the optimal distribution as

$$\mathcal{E}[\rho_*(x;s)] = \frac{\mu_1}{2}\left(\frac{1}{x_0} - s\right) + \int_a^b \mathrm{d}x\,\rho_*(x;s) \times \left[\frac{x - \ln x + x_0 - \ln x_0}{2} - \ln|x-x_0|\right]\,. \quad (11)$$

Optimal distribution.– The integral equation (9) may be solved thanks to a theorem due to Tricomi [35]. We find the optimal distribution

$$\rho_*(x;s) = \frac{1}{2\pi}\frac{x+c}{x^2}\sqrt{(x-a)(b-x)}\,, \quad (12)$$

where the three parameters a, b and $c = \mu_1/\sqrt{ab}$ can be found by solving the three algebraic equations obtained by imposing the vanishing of the density at the two boundaries and the condition $\int_a^b \frac{\mathrm{d}x}{x}\,\rho_*(x;s) = s$. These equations are conveniently written in terms of the variables $v = \sqrt{ab}$ and $u = \sqrt{a/b}$. A few steps of algebra shows that u solves

$$s = \sigma(u) \overset{\text{def}}{=} (1-u)^2\,\frac{(-u^4 + 16u^3 + 2u^2 + 16u - 1)}{16u^2(3u^2 - 2u + 3)}\,. \quad (13)$$

Then v, μ_1 and c are given by $v = 2u\,\frac{3u^2-2u+3}{(1-u^2)^2}$, $\mu_1 = -4u^2\,\frac{(u^2-6u+1)(3u^2-2u+3)}{(1-u^2)^4}$ and $c = \frac{\mu_1}{v} = -2u\,\frac{(u^2-6u+1)}{(1-u^2)^2}$.

Most probable values.– We first analyse the distribution $P_N(\tau)$ in the vicinity of its maximum. $\mathcal{E}[\rho_*(x;s)]$ is minimised, i.e. $P_N(\tau)$ is maximized, by removing the constraint $\int_a^b \frac{\mathrm{d}x}{x}\,\rho(x) = s$, i.e. by setting $\mu_1 = 0$. For convenience we introduce the roots $x_\pm = 3 \pm 2\sqrt{2}$ of the polynomial u^2-6u+1. For $\mu_1 = 0$, Eq. (13) has solution $u = \sqrt{x_-/x_+} = x_-$ with $v = 1$ and $s = 1$ and consequently $a = x_- = 0.171...$ and $b = x_+ = 5.828...$ In this case we recover the Marčenko-Pastur (MP) law [36]

$$\rho_*(x;1) = \frac{1}{2\pi x}\sqrt{(x-x_-)(x_+-x)} \tag{14}$$

Expansion of Eq. (13) around the MP point leads to $s-1 \simeq -\frac{x_+}{\sqrt{2}}(u-x_-)$, hence $v \simeq 1 + \frac{3x_+}{2\sqrt{2}}(u-x_-) \simeq 1 - \frac{3}{2}(s-1)$ and $c \simeq \mu_1 \simeq -\frac{1}{2}(s-1)$. The corresponding energy (11) may be conveniently obtained by choosing $x_0 = 1$: we see that the first term is quadratic $\frac{1}{4}(s-1)^2$; we check numerically that the remaining integral term is constant, equal to $\mathcal{E}[\rho_*(x;1)] = 3 - 2\ln 2$, up to higher order corrections [numerical fit gives a correction $\frac{1}{4}(s-1)^3$]. Therefore we conclude that $\Phi_-(s) \underset{s\sim 1}{\simeq} \frac{1}{4}(s-1)^2$, i.e. Eq. (3) (the parabolic behaviour is compared to the numerical calculation of the integral (11) in Fig. 3).

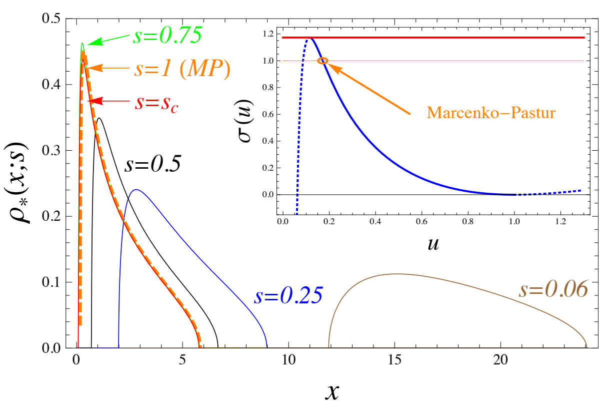

FIG. 2: (color online) *The optimal density of eigenvalues for different values of s ; when s increases, the density eventually freezes to the MP law (dashed line).*

Large deviations for $s \to 0$.– Expansion of (13) for $s \to 0$ gives $u = 1 - \sqrt{2s} + s + \mathcal{O}(s^{3/2})$, hence $v = \frac{1}{s} + \mathcal{O}(s^0)$. The support of the distribution is given by $a = \frac{1}{s}\big[1 - \sqrt{2s} + \mathcal{O}(s)\big]$ and $b = \frac{1}{s}\big[1 + \sqrt{2s} + \mathcal{O}(s)\big]$ (the Lagrange multiplier is $\mu_1 = \frac{1}{s^2} + \mathcal{O}(s^{-1})$). The optimal distribution ressembles the semi-circle law centered around $1/s$:

$$\rho_*(x;s) \underset{s\to 0}{\simeq} \frac{1}{\pi}\sqrt{2s - (s\,x-1)^2}\,. \tag{15}$$

This was expected : when $s \to 0$, the eigenvalues $\{x_i\}$ of the Wishart matrix are constrained to be very large and they do not feel the spectrum boundary at $x = 0$. Hence, their distribution coincides with the Wigner semi-circle law for the usual Gaussian ensembles of random matrices. The energy may be conveniently calculated by choosing $x_0 = 1/s$; this makes the first term of (11) vanish. The leading order of the integral term is straightforwardly calculated from (15) : we deduce $\Phi_-(s) \underset{s\to 0}{\simeq} \frac{1}{s} + \frac{3}{2}\ln s - \frac{5}{2}(1-\ln 2)$, thus proving (2). The factor $\exp -\frac{N\beta}{2\tau}$ is in perfect agreement with the exact results for $N = 1$ & 2 mentioned earlier.

Large deviations for $s \geqslant 1$ – Freezing transition.– As s increases, it eventually reaches a finite value corresponding to the maximum of the function $\sigma(u)$ (inset of Fig. 2), at $u_c = \frac{1}{3}\big[1 + 2(2^{1/3} - 2^{2/3})\big] = 0.115...$ giving $s_c = \sigma(u_c) = \frac{10+6\times 2^{1/3}-11\times 2^{2/3}}{3\,(6-6\times 2^{1/3}+2^{2/3})} = 1.1738....$ Then $a = -c$, which leads to a somewhat unusual form

$$\rho_*(x;s_c) = \frac{1}{2\pi x^2}(x-a)^{3/2}(b-x)^{1/2}\,. \tag{16}$$

For $s > s_c$, (13) has no longer physical (real) solutions. In this case, the saddle point turns out to have a different solution where a single isolated charge, say at x_1, splits off the main body of the density and carries a macroscopic weight (see Fig. 1). A similar scenario occurs in the study of quantum entanglement in random bipartite state [31–33]. We decompose the density as $\rho(x) = \frac{1}{N}\delta(x-x_1) + \tilde{\rho}(x)$ where $\tilde{\rho}(x) = \frac{1}{N}\sum_{i>1}\delta(x-x_i)$ is still treated as a continuous density. The energy

$$\mathcal{E}[\rho] = \mathcal{E}[\tilde{\rho}] + \frac{x_1 - \ln x_1}{N} - \frac{2}{N}\int \mathrm{d}x\,\tilde{\rho}(x)\,\ln(x-x_1) \tag{17}$$

must be minimized under the two constraints $\int \mathrm{d}x\,\tilde{\rho}(x) = 1 - \frac{1}{N}$ and $\int \mathrm{d}x\,\frac{\tilde{\rho}(x)}{x} = s - \frac{1}{Nx_1}$. This leads to the two equilibrium conditions

$$\frac{1}{2}\left(1 - \frac{1}{x_0} - \frac{\mu_1}{x_0^2}\right) - \frac{1}{N}\frac{1}{x_0 - x_1} = \fint_a^b \mathrm{d}x'\,\frac{\tilde{\rho}(x')}{x_0 - x'} \tag{18}$$

$$\frac{1}{2}\left(1 - \frac{1}{x_1} - \frac{\mu_1}{x_1^2}\right) = \int_a^b \mathrm{d}x'\,\frac{\tilde{\rho}(x')}{x_1 - x'}\,, \tag{19}$$

$\forall\, x_0 \in [a,b]$ and $x_1 < a$. We show that a consistent picture is the freezing of the density $\tilde{\rho}(x)$ while the isolated charge goes to zero $x_1 \to 0$. When $N \to \infty$, the r.h.s. of (19) reaches a constant value as $x_1 \to 0$; so does the l.h.s. iff $\mu_1 \simeq -x_1 \to 0^-$. Hence the solution of (18) is the MP law : $\tilde{\rho}_*(x;s) = \rho_*(x;1) + \mathcal{O}(N^{-1})$. The rescaled time delay splits into the contribution of the isolated charge and of $\tilde{\rho}$ as $s = \frac{1}{Nx_1} + 1$, i.e. $x_1 = 1/[N(s-1)]$. In fact this analysis holds for any $s > 1$ (and not only $s \geqslant s_c$) : the energy (17) of this new phase coincides with the energy of the MP solution, up to $1/N$ corrections. Therefore for $1 < s \leqslant s_c$ we have found another phase with a lower energy, which shows that the branch obtained previously (with compact solution (12) over $[a,b]$ for $s < s_c$ as well

as (16) for $s = s_c$) is actually *metastable* (Fig. 3). In the (thermodynamic) limit $N \to \infty$, the energy of the gas vanishes for all $s > 1$, while for $s < 1$, it behaves as $\frac{1}{4}(1-s)^2$ as mentioned earlier (Fig. 3). This then results in a *second order* phase transition at $s = 1$. We call this a *freezing* transition, because for $s > 1$, energy freezes to the value 0 in the thermodynamic limit and also the bulk density freezes to the MP distribution.

One can analyse more precisely this new *frozen* phase by computing the $1/N$ corrections to the energy. For large enough s, Eq. (17) is dominated by the logarithmic term $-\frac{1}{N}\ln x_1$, i.e. $\mathcal{E}[\rho_*(x;s)] \simeq (\cdots) + \frac{1}{N}\ln\left[N(s-1)\right]$. We get the power law tail $P_N(\tau) \sim (s-1)^{-\tilde{\theta}-\frac{\beta}{2}N}$, where $\tilde{\theta}$ is some exponent of order N^0 introduced in order to account for N^{-2} corrections to $\mathcal{E}[\rho]$. This exponent may be determined as follows : when $\tau_{\rm W} > 1/N$, most of the proper times are described by the frozen density (the MP law), i.e. $\tau_i \in [x_-/N, x_+/N]$ for $i > 1$ with $\sum_{i>1} 1/\tau_i = 1$, while one proper time becomes much larger and carries a "macroscopic" contribution, $\tau_1 = s - 1 = N\tau_{\rm W} - 1$. In the scattering problem, this is interpreted as the large contribution of a narrow *resonance*. Writing $P_N(\tau) = \int \mathrm{d}\gamma_1 \cdots \mathrm{d}\gamma_N\, \delta(N\tau - 1/\gamma_1 - 1)\, P(\gamma_1, \cdots, \gamma_N)$ and using (1) leads to $\tilde{\theta} = 2$, hence Eq. (5).

A more precise analysis of Eqs. (18,19) leads to introduce the large deviation function $\Phi_+(s) = N\left(\mathcal{E}[\rho_*(x;s)] - \mathcal{E}[\rho_*(x;1)]\right) - \ln N$ giving the scaling form

$$P_N(\tau) \sim N^{-\frac{\beta N}{2}} \exp -\frac{\beta N}{2}\Phi_+(N\tau) \quad \text{for } \tau > \frac{s_N}{N} \qquad (20)$$

One obtains that $\Phi_+(s) = \frac{1}{s-1} + \ln(s-1) - 1 - 2\ln 2$ (c.f. inset of Fig. 3). The local minimum at $s = 2$ is related to (4) while the logarithmic behaviour to the power law tail (5). For finite N, the energy functions characterising the two phases cross for $s = s_N$ such that $\Phi_-(s_N) = \frac{1}{N}\left[\Phi_+(s_N) + \ln N\right]$. Using the limiting behaviours for $s \to 1$, we obtain the finite N correction to the position of the phase transition : $s_N \simeq 1 + (4/N)^{1/3}$.

Conclusion.– In summary, by using a Coulomb gas approach, we have analysed the large deviation functions controlling the Wigner time-delay distribution in the limit of large number of conducting channels. We have shown that the distribution exhibits a rich structure. In particular, its power law tail is related to a freezing transition in the Coulomb gas, corresponding to large contributions to $\tau_{\rm W}$ of resonant states in the original scattering problem. We have also performed a Monte-Carlo simulation of the Coulomb gas up to 1600 charges and found good agreement with our analytical results (details will be published elsewhere).

Several questions remain open : (*i*) a more precise treatment of $1/N$ corrections would be desirable. (*ii*) The starting point of our calculation, Eq. (1), describes the usual random matrix ensembles ; the distribution of $\tau_{\rm W}$ was also obtained in [37] for a chiral-GUE ensemble when $N = 1$. Extension of our analysis to such cases would be certainly interesting, in particular with the growing interest in the study of new symmetry classes of disordered systems.

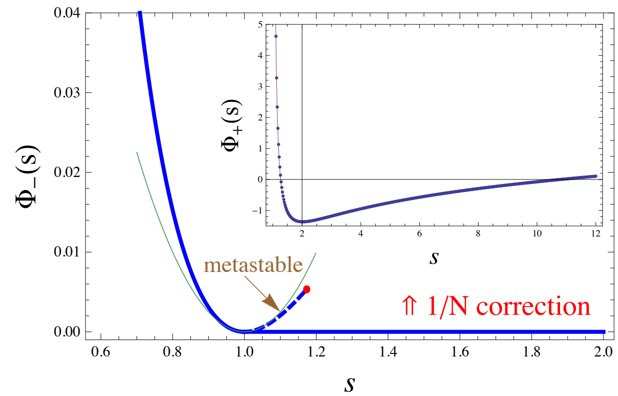

FIG. 3: (color online). *Large deviation function $\Phi_-(s)$ (i.e. rescaled energy of the gas). The freezing transition takes place at $s_\infty = 1$. The metastable branch terminates at $s_c = 1.1738...$ Inset : Large deviation function $\Phi_+(s)$ [i.e. $1/N$ correction to the rescaled energy].*

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[14] The partial time delays $\tilde{\tau}_a$ are defined as derivatives of the phase shifts [phases of the eigenvalues of $\mathcal{S}(E)$]. $\tilde{\tau}_a$ measures the time spent in the scattering region by a wave packet in scattering channel a with narrow dispersion in energy around E. Because derivation and diagonalisation do not commute, partial times $\tilde{\tau}_a$ differ from proper times τ_a ; they however satisfy the sum rule $\sum_a \tilde{\tau}_a = \sum_a \tau_a$.

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